THE CATENARY AND HYPERBOLIC FUNCTIONS

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Abstract. A uniform perfectly flexible chain hangs freely from two fixed points in a catenary, a curve characterized by a hyperbolic cosine function. This can be shown using elementary calculus, but it also raises the question about how to compute the integrals involved if you know some basic calculus facts but don't know about the hyperbolic functions, which was the situation when the problem was first solved. I solve the problem using modern calculus techniques both with and without assuming prior knowledge of the hyperbolic functions. I conclude by noting how it was done originally when the methods of calculus were first being developed on the European continent by Leibniz and the Bernoullis, and I sketch some history of the hyperbolic functions.

1. Introduction

Galileo, not having calculus, could not have determined the exact shape of a hanging chain, but he could guess shrewdly. He was wrong in one instance, but not off by much, when he wrote in *Two New Sciences* that a freely hanging chain "will assume the form of a parabola." Robert Osserman [11] balances the record in an article explaining the mechanics of the Gateway Arch, where he finds that Galileo also wrote that "...a cord stretched more or less tightly assumes a curve which *closely approximates* the parabola....[emphasis added]". Osserman illustrates how close to within visual accuracy the parabola and catenary can be.

Surely it was inevitable that the early practitioners of calculus would take up the problem of the hanging chain. Jakob Bernoulli posed the problem as a challenge to his fellow mathematicians in *Acta Eruditorum* of 1690. None other than Leibniz, Huygens and Johann Bernoulli responded with solutions in the following year, 1691. Huygens had given the name *catenary* to the hanging chain, and Leibniz and Johann

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Acknowledgements. The catenary problem returned to my attention when reading an advance copy of an article about the Gateway Arch, cited below, for which I thank the author Robert Osserman. I also want to thank Janet Barnett for conversations about her amazing history of the hyperbolic functions, cited below. I alone am responsible for interpretations of their work.

Bernoulli deduced differential equations for their solutions. What is surprising is that all three expressed their catenaries as geometric constructions, not as explicit exponential functions, let alone as hyperbolic functions. These latter functions had not yet entered the vocabulary of mathematics: "Enter, Stage Center: The Early Drama of the Hyperbolic Functions" [2].

Before knowing about the original work on the catenary or Barnett's history, I considered: What if I didn't know about the hyperbolic functions, how could I find an analytic solution for the hanging chain? Try it yourself! It's an interesting problem.

With hindsight of course and tools of modern calculus in hand, my answer led straight back to the *power series* for the hyperbolic functions. This turned out very different from the earliest work, which I sketch along with Barnett's discoveries in the Conclusion where I invite you to help fill in the picture.

But now, on to a solution for the catenary problem without prior knowledge of the hyperbolic functions!

2. Setting up the equation

Let a freely-hanging flexible chain of uniform density ρ pounds per unit length be parametrized by arc length, $\mathbf{x}(s)$. Let $\tau(s)$ be the *tension* within the chain, a value that we stipulate as positive characterizing the forces acting within the chain and aligned everywhere with the tangent along the chain. Since the unit tangent vector at $\mathbf{x}(s)$ is given by $\dot{\mathbf{x}}(s)$, the tangential force at a point can be expressed as $\pm\tau(s)\dot{\mathbf{x}}(s)$, a force that acts equally and oppositely at every point within a chain at equilibrium.

Consider a segment of the chain located between $\mathbf{x}(s-\delta)$ and $\mathbf{x}(s+\delta)$. Because the segment is in equilibrium, the net external forces acting on the segment must be null. Those forces are the ones conveyed tangentially by tension at the endpoints, as well as by the net force of gravity acting vertically on the segment. Expressing the mutual cancellation of forces as an equation,

$$\tau(s+\delta)\dot{\mathbf{x}}(s+\delta) - \tau(s-\delta)\dot{\mathbf{x}}(s-\delta) + (0,0,-2\rho\delta) = 0 \tag{1}$$

Therefore,

$$\frac{\tau(s+\delta)\dot{\mathbf{x}}(s+\delta) - \tau(s-\delta)\dot{\mathbf{x}}(s-\delta)}{2\delta} = (0,0,\rho) = 0$$

Taking the limit,

$$\frac{d}{ds}[\tau(s)\dot{\mathbf{x}}(s)] = (0,0,\rho)$$

and integrating,

$$\tau(s)\dot{\mathbf{x}}(s) = (a, b, c+\rho s)$$

for constants a, b, c.

Assuming that the curve is horizontal at $s = s_0$ with the unit vector pointing in the direction of the positive x-axis, i.e., $\dot{\mathbf{x}}(s_0) = (1,0,0)$, we find,

$$a = \tau(s_0);\ b = c + \rho s_0 = 0$$

and conclude,

$$\tau(s) = ||\tau(s)\dot{\mathbf{x}}(s)|| = \sqrt{\tau_0^2 + \rho^2(s-s_0)^2}$$

where the positive sign of the square root was chosen to ensure that tension is positive.

Now we may write,

$$\tau(s)\dot{\mathbf{x}}(s) = (\tau_0, 0, \rho(s-s_0)) \tag{2}$$

and,

$$\dot{\mathbf{x}}(s) = \left(\frac{\tau_0}{\tau(s)}, 0, \frac{\rho(s-s_0)}{\tau(s)}\right) \tag{3}$$

Eq. (2) shows that the horizontal component of tension is independent of position $\mathbf{x}(s)$ and that the vertical component of tension at $\mathbf{x}(s)$ equals the weight of the chain between $\mathbf{x}(s)$ and the low point of the chain at $\mathbf{x}(s_0)$. Eq. (3) shows that the chain hangs in the vertical x-z plane; we will have to integrate it to find a formula for the graph of the chain.

3. Solving the equation

The notation $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$ is used interchangeably. Eq. (3) can be solved by changing variables to find $z = x_3$ as a function of $x = x_1$:

$$\frac{dz}{dx} = \frac{\dot{x_3}}{\dot{x_1}} = \frac{\rho}{\tau_0}(s - s_0)$$

Using "prime" ($'$) to indicate differentiation by x, note that $s' = \sqrt{1+z'^2}$, so differentiating the foregoing equation wrt x yields the equation we must solve,

$$z'' = \frac{\rho}{\tau_0}\sqrt{1+z'^2} \tag{4}$$

3.1. **Evaluating the integrals using hyperbolic functions.** Substituting $w = z'$ in Eq. (4) we arrive at an integration problem,

$$\int \frac{dw}{\sqrt{1+w^2}} = \frac{\rho}{\tau_0}x + C_0 \tag{5}$$

Jumping ahead, I digress briefly to remind ourselves how easy it is to solve this equation using present-day knowledge. Note that the integral on the left-hand side of Eq. (5) can be evaluated using a hyperbolic function, the usage I foreswore. Set $w = \sinh t$ in the integrand to find that,[1]

$$\operatorname{arcsinh} w = \frac{\rho}{\tau_0}x + C_0$$

Inverting the previous equation,

$$w = z' = \sinh\left(\frac{\rho}{\tau_0}x + C_0\right)$$

Assume that the low point of the chain is at $x = 0$, in which case $C_0 = 0$. Using that assumption and integrating the previous equation,

$$z = \frac{\tau_0}{\rho}\cosh\left(\frac{\rho}{\tau_0}x\right) + C_1 \tag{6}$$

Eq. (6) gives our final result: the *catenary* can be expressed simply in terms of a hyperbolic cosine.[2] Because of the straightforward integration and neatness of the result, you might suspect that the hyperbolic

[1] It is at this point in the analysis that the question arises: What if you don't have the hyperbolic functions? That will be the problem dealt with in the next two subsections.

[2] A similar derivation is given in Redheffer [12, 105–107]. Hairer and Wanner [6, pp 135–136] set up one of Johann Bernoulli's differential equations for the catenary and solve it non-historically using hyperbolic functions.

functions were at hand for the original solutions of the catenary problem, but we shall see in the history discussion in the Conclusion that this is far from true.

4. Alternative approach to the hyperbolic sine

But now let's renew our vow not to use the hyperbolic functions and ask, How can we find a convenient form for the integral of Eq. (5)? Here's a way to do it.

Suppose we already know about the trigonometric functions and are familiar with the integral $\int \frac{dt}{\sqrt{1-t^2}} = \arcsin x + C$. This might suggest that the integral of Eq. (5) is an inverse function of a nice kind, comparable to the arcsin—a wild guess perhaps but you can't know without trying.

So suppose $g(x)$ is an inverse function such that

$$x = \int_0^{g(x)} \frac{dt}{\sqrt{1+t^2}} \tag{7}$$

Differentiating wrt x yields,[3]

$$1 = \frac{g'(x)}{\sqrt{1+g(x)^2}} \tag{8}$$

Or

$$g'(x)^2 = 1 + g(x)^2$$

We're motivated by the example of the sine function. Letting,

$$g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

we hope, as in the case of $\sin x$, to find an elegant pattern for the coefficients. Substitution in Eq. (8) gives:

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[3]We could as easily and directly reach an equivalent equation for $g(w)$ by substituting $z' = g(w)$ in (4), where $w = (\rho x/\tau_0)$. But the "sine integral" analogy helps to motivate expectation of an elegant result. For a classic treatment of differentiating integrals wrt to functions in the limits or with respect to a free variable in the integrand see Courant and John [4].

$$\left(\sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} x^{n-1}\right)^2 = \left(\sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} x^n\right)^2 = 1 + \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right)^2 \tag{9}$$

This leads us to an exercise in equating coefficients of like powers to determine the coefficients progressively. Begin by collecting coefficients of x^0 to get,

$$a_1^2 = 1 + a_0^2$$

To simplify, let $a_0 = 0$ and get $a_1 = 1$ (we could have selected $a_1 = -1$, but we're starting with the simplest choices). Now collect coefficients of x^1,

$$2a_1a_2 = 2a_0a_1 \implies a_2 = 0$$

Collecting coefficients of x^2,

$$a_1\frac{a_3}{2!} + a_2a_2 + \frac{a_3}{2!}a_1 = a_0\frac{a_2}{2!} + a_1a_1 + \frac{a_2}{2!}a_0$$

and substituting known values yields,

$$a_3 = 1$$

Then going two steps farther, we find,

$$a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = 1,\ a_4 = 0,\ a_5 = 1$$

This leads to a presumption that $a_{2n} = 0, a_{2n+1} = 1$ for all $n = \mathbb{N}_{\geq 0}$, and to a test of it by mathematical induction. But there is another option as well—a shortcut. Our problem is just to find the inverse function of the integral in Eq (5), and what we have in view now is a candidate solution in the form of the infinite series comprised of the odd powers of the exponential function. Denote the candidate solution temporarily as,

$$E_{odd}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \tag{10}$$

We can either try proving by mathematical induction that $g(x) = E_{odd}(x)$, or we can take a shortcut by playing with this pretendedly new function E_{odd} to test directly whether it solves our problem. Strictly speaking, mathematical induction is unnecessary for our purpose, so if you like shortcuts, you can skip this next bit of work and jump ahead to the next section.

But since we're on the path of induction, a nice exercise, I'll follow through with it first. Begin by expanding the squares in Eq (9). First, the left-hand side,

$$\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{a_{i+1}}{i!}\frac{a_{n+1-i}}{(n-i)!}\right)x^n=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}a_{i+1}a_{n+1-i}\right)\frac{x^n}{n!}$$

Then the right-hand side,

$$1+\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}a_ia_{n-i}\right)\frac{x^n}{n!}$$

Equating coefficients for $n>0$,

$$\sum_{i=0}^{n}\binom{n}{i}a_{i+1}a_{n+1-i}=\sum_{i=0}^{n}\binom{n}{i}a_ia_{n-i} \tag{11}$$

Assume the inductive hypothesis for $i\le n$, i.e., for i even $a_i=0$ and for i odd $a_i=1$. The case for odd n is immediate because the right-hand side of Eq. (11) vanishes, and the left-hand reduces to $2a_{n+1}$, showing the $a_{n+1}=0$. Here n is odd so $n+1$ is even.

The case for an even positive integer n is more involved. In this case Eq. (11) reduces to

$$a_{n+1}+\sum_{i=1}^{n/2-1}\binom{n}{2i}+a_{n+1}=\sum_{i=0}^{n/2-1}\binom{n}{2i+1} \tag{12}$$

But,

$$\binom{n}{0}+\sum_{i=1}^{n/2-1}\binom{n}{2i}+\binom{n}{n}=\sum_{i=0}^{n/2}\binom{n}{2i}$$

Therefore, Eq. (12) can be rewritten as,

$$2a_{n+1} = 2 - \left(\sum_{i=0}^{n/2} \binom{n}{2i} - \sum_{i=0}^{n/2-1} \binom{n}{2i+1} \right) = 2 - (1-1)^n = 2$$

Thus, for n even—and $n+1$ odd—we infer that $a_{n+1} = 1$.

This proves that for $n \in \mathbb{N}_{\geq 0}$, $a_n = 0$ if n is even, and $a_n = 1$ if n is odd, as surmised. In other words, the inverse function g of Eq. (7) is given by the power series for $E_{odd}(x)$ of Eq. (10), which of course we know as the *hyperbolic sine function*:

$$g(x) = E_{odd}(x) = \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)!} = \frac{e^x - x^{-x}}{2} = \sinh x$$

4.1. **The shortcut starts here.** Having arrived at a power series for $g(x)$ comprising the odd-power terms in the expansion for e^x, known as $\sinh x$, it is natural to examine the series of complementary terms,

$$E_{even}(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{e^x + e^{-x}}{2} = \cosh x$$

It is now a routine exercise to find formulas and identities analogous to those for the trigonometric functions. The relevant facts are:

$$\frac{d}{dx} \cosh x = \sinh x; \ \frac{d}{dx} \sinh x = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \operatorname{arcsinh} x + C$$

The preceding antiderivative is what allowed us to integrate Eq. (5); it is derived by substitution from the two lines before it. And the first of the two derivative formulas with its implicit antiderivative was used to derive the function $z(x)$ in Eq. (6) from the preceding equation for $z'(x)$. So in our round-about way we have reached our goal. We have solved the catenary problem without foreknowledge of the hyperbolic functions.

See Apostol [1, Secs. 6.18, 14.7 Prob. 12] for an introduction to the hyperbolic functions, including a discussion of their relevance to the hyperbola and analogy with the trigonometric functions.[4]

5. Conclusion

What began as a casual exercise in calculus led to considering how to derive the equation for a catenary without assuming prior knowledge of the hyperbolic functions. So, "I did it my way."

But that led to the question, How in fact was the problem solved originally? I noted in the Introduction that the first solvers succeeded without explicit use of hyperbolic functions—they hadn't been invented yet [2]. Instead, they stated explicitly how to *construct* a catenary curve.

Here I sketch my understanding based on English translations of the methods used by Leibniz and Johann Bernoulli to solve the problem. I don't know how Huygens did it. If you know more about these constructions, please let me know.

Leibniz. Leibniz based his construction on his "logarithmic curve". I infer his construction from the figure and explanation in [10]. What he constructs is actually an exponential curve, but he views it sideways to obtain logarithms. The idea is that between two given points with positive ordinates, he determines the ordinate of the midpoint by the geometric mean of the two given ordinates. Expressing this generically in XY coordinates, not as Leibniz did, suppose given (x_1, y_1) and (x_2, y_2), then the coordinates of the midpoint must be,

$$(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \sqrt{y_1 y_2}\right) \tag{13}$$

A curve is then propagated by repeating the rule for midpoints as often as you like. Using the starting values posited by Leibniz, which expressed in our XY-coordinates are $(0, K)$ and $(-K, D)$, and extending the function by continuity, we may deduce in general,[5]

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[4]For example, note that $(x, y) = (\cosh t, \sinh t)$ gives a parametric representation of a hyperbola just as $(x, y) = (\cos t, \sin t)$ represents a circle. The analogy goes deeper, as seen in Apostol and more so in Barnett. For example, an angle in trigonometry is equivalent to the area swept out by the angle in a unit circle. An analogous angle and area can be defined for hyperbolic functions.

[5]The formula can be inferred by using Eq (13) to find a few intermediate values, such as, say, $y(-K/2)$ and $y(-3K/4)$, then extending the domain by using, say, $y(-K)$ and $y(0)$ to infer $y(+K)$, and $y(-2K)$, and so forth. Verification of Eq.

$$y = K \left(\frac{K}{D} \right)^{\frac{x}{K}} \tag{14}$$

Leibniz obtains a class of curves by averaging the ordinates of two symmetric points about the origin of this curve, i.e., by taking the mean value of exponentials, and assigning the mean as ordinates for two symmetric points of his catenary. This results in curves, dependent on parameters K and D, that in our coordinates are represented as,

$$y = \frac{K}{2} \left[\left(\frac{K}{D} \right)^{\frac{x}{K}} + \left(\frac{K}{D} \right)^{-\frac{x}{K}} \right]$$

Comparison with Eq. (6) shows that this equation does not yield a true catenary unless Leibniz has arranged in effect that $\frac{K}{D} = e$,[6] but it is not apparent that he has done that in the cited exposition, although he alludes to prior work using his differential calculus in which he may have been more specific.[7]

Bernoulli. Johann Bernoulli, in setting up differential equations for the catenary, does provide a scheme for producing a true catenary and concludes that the resulting curve is of the class described by Leibniz.

Bernoulli [3] begins with a heuristic discussion[8] using differentials to characterize the tangential and gravitational forces acting on the hanging chain. He argues for a balance of forces equivalent to the one expressed by Eq. (1) and using it to arrive at a differential equation. I copy his equation literally except for interchanging his variables x and

(14) in general follows by noting its correctness for $y(0)$ and $y(-K)$ and showing that it satisfies Eq. (13) for the midpoint of any two arbitrary points.

[6]I use "e" here in to denote the base of the natural logarithm, a practice not common until the time of Euler.

[7]"[T]he well known [Jacob] Bernoulli...asked me publicly...if I would examine the problem of the catenary curve , and see if with our calculus, I could come up with a determination of the curve...unless I am mistaken, I have...succeeded in becoming the first to solve this famous problem....Here is a geometric construction for the curve, without the use of a string, and without using any chain, and without any assumption of a quadrature...." At one point in his construction Leibniz writes that it is "child's play"! Check the reference to see if you agree.

[8]"This needs no proof, because Resason advises it and experience lays it daily before our eyes."

y for the convenience of using present-day notation for horizontal and vertical coordinates, opposite to Bernoulli's usage,

$$dx = \frac{a dy}{\sqrt{2ay + y^2}}$$

Bernoulli does not integrate this equation to find a functional representation but instead continues an intricate analysis to show how the equation yields solution curves of the kind constructed by Leibniz. But we can short-circuit all of that by simply integrating the equation: for example, by completing the square in the denominator, and substituting $y + a = a \cosh w$, or by table lookup to find an integral in terms of the natural logarithm (cf, [9, Sec. 4.3.3]), either way arriving at the solution,

$$x + c = a \cosh^{-1} \frac{y + a}{a} \implies y = -1 + a \cosh \frac{x + c}{a}$$

Comparison with Eq. (6) shows that Bernoulli's equation does in fact yield a true catenary, but one that is shifted differently.[9]

Hyperbolic functions. The geometric constructions described by Leibniz and Bernoulli are reminiscent of the geometric methods used by Newton in his *Principia*. While common at the time, they now can seem laborious to readers who find it more convenient to deal directly in terms of functions.

So when did the hyperbolic functions come into use? In her study of the rise of the hyperbolic functions [2], Barnett writes,[10]

> ...one naturally looks to the works of Euler. In fact,the expressions $(e^x+e^{-x})/2$ and $(e^x-e^{-x})/2$ did make an appearance in Volume I of Euler's *Introductio in analysin infinitorum* (E101) 1748.... However, Euler's interest in what we would call hyperbolic functions appears to have been limited in deriving infinite product representations for the sine and cosine functions. Euler did not use the word *hyperbolic*....[11]

[9]Jahnke [8, Sec. 4.2] provides a very brief account of Bernoulli's approach.

[10]Except where noted otherwise, I draw on Barnett for information about Lambert.

[11]This is not to say that Euler was disinterested in the catenary. His discovery that the *catenoid*, a surface swept out by spinning a catenary around its abscissa, had minimal surface area for the surface subtended between its bounding circles, was among the first applications of his calculus of variations and marked the beginning of the theory of minimal surfaces. [5, Lützen, 213]

According to Barnett, Johann Heinrich Lambert "first treated hyperbolic trigonometric functions in a paper presented to the Berlin Academy of Sciences in 1761 that quickly became famous: *Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques.*"[12] Lambert compared the "transcendantes circulaire", i.e., $\sin v$ and $\cos v$, with their analogues, the "quantitiés transcendantes logarithmiques", $(e^v + e^{-v})/2$ and $(e^v - e^{-v})/2$, functions which he dealt with explicitly in their functional form and as power series, but did not yet name them the hyperbolic sine and cosine. That came later in his *Observations trigonométriques* of 1768 in which he makes a comparison of quotients of trigonometric and hyperbolic functions,

$$\frac{\sin \phi}{\cos \phi} \quad \text{and} \quad \frac{\sin hyp\, \phi}{\cos hyp\, \phi}$$

Barnett continues, "Although Lambert's notation for these functions differed from our current convention, the hyperbolic functions had now become fully-fledged players in their own right, complete with names and notation suggestive of their relation to the circular trigonometric functions."

A curious twist in the story is that, although Lambert is often credited with being the first to introduce the hyperbolic functions, he was not. That credit is due to Vincenzo de Riccati, son of Jacopo Riccati, namesake of the Riccati equation. Vincenzo's ideas appeared in his *Opuscula ad res physicas et mathematicas pertinentium* of 1757–1762, where he used the hyperbola to define the "sinus hyperbolico" and "cosinus hyperbolico" similar to the way in which a circle is used to define the corresponding trigonometric terms.[13] According to Barnett, Lambert was always generous in attribution, and there is no evidence that either Riccati or Lambert derived his results from the other. Lambert does credit Riccati with having introduced the terms "hyperbolic sine" and "hyperbolic cosine".

So why did Lambert get the credit? Lambert's work was better known to later mathematicians than Riccati's. In addition to Lambert's achievements documented by Barnett, including the famous proof of the irrationality of π, extensive work on the hyperbolic functions, and a table of hyperbolic functions for solving astronomical triangles related to contemporaneous efforts that led eventually to formulations

[12]The paper became famous not because of the hyperbolic functions but because in it Lambert presented the first explicit proof that π is irrational.

[13]The correspondence is explained in Apostol, cited above.

of noneuclidean geometry, Lambert is also known for other innovations. Along with Euler and Lagrange, Lambert published seminal work in cartography [7]:

> Lambert's work is considered the foundation of modern mathematical cartography. Lagrange gives Lambert credit as the first to characterize the problem of mapping from a sphere to a plane, while preserving a given property, in terms of nonlinear partial differential equations. The technique was astonishingly fruitful, for he invented several families of conformal and equal-area projections, some of which are still in widespread use today. Lambert also seems to have been the first to take account of the ellipsoidal, rather than spherical, shape of the earth.... [Lambert's Transverse Mercator Projection,] with adjustments for the ellipsoidal shape of the earth (also given by Lambert) is today the most widely used projection for large scale maps in the United States."

Apparently the catenary had nothing to do with Lambert's interest in the hyperbolic functions, which sprang instead from concerns with trigonometric analogies and the applicability of hyperbolic functions to noneuclidean geometry. Barnett writes [2, 100-101]:

> In short, despite the fact that his interests often fell outside the mainstream of his own century, the motivation which Lambert provided for the hyperbolic functions was more central to mathematical interests as they later evolved than that provided by Riccati. The fact that Lambert's mathematical works, especially those on noneuclidean geometry, were studied by his immediate mathematical successors offers support for this idea, as does the availability of Lambert's work today. Lambert's work is written in notation—and languages!—that are more familiar to today's scholars than that of Ricatti. This alone makes it easier to tell Lambert's story in detail....

One final question. What began here as a calculus exercise resulted in a question about the history of the hyperbolic functions. Mathematical methods, like words, categories and concepts, evolve from tangled webs of human interactions. We solve the catenary problem as a routine calculus exercise, typically using properties of hyperbolic functions, but this is foreign to the methods used by the original solvers. Now I am wondering: Who was it that first solved the catenary problem by making explicit use of hyperbolic functions and expressing the result as a hyperbolic cosine?

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